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A note on nearly two-dimensional weakly nonlinear instability of an incompressible free shear layer

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The weakly nonlinear amplitude growth of slightly oblique instability waves in an incompressible free shear layer is shown to be first influenced by three-dimensionality in a limiting case for large Reynolds number when a particular order relationship is chosen between the spanwise scale and the amplitude of the small disturbance. The formulation resembles that for purely two-dimensional motion but includes the effect of vortex stretching in the nonequilibrium, nonlinear, viscous critical layer.

1. Introduction

The instability of a two-dimensional incompressible free shear layer has been the subject of numerous studies over many years. Of particular concern here is a recent series of papers containing self-consistent asymptotic descriptions of the weakly nonlinear spatial growth of small disturbances to the shear layer. Goldstein & Leib (1988), Goldstein & Hultgren (1988), and Hultgren (1992) considered two-dimensional disturbances, while Goldstein & Choi (1989) described the growth of a three-dimensional disturbance, for a simple harmonic spanwise variation, i.e. for a pair of oblique waves having the same amplitude and frequency, travelling at equal but opposite angles from the direction of the mean flow (and thus representing a standing wave in the spanwise direction). Hereafter these references will be referred to as GL, GH, H, and GC, respectively. A hyperbolic tangent velocity profile was assumed in GL and GH; a more general profile was allowed in H and GC. Inviscid flow was considered in GL and GC; the effect of small viscosity was included in GH and H, and in an extension of GC given by Wu, Lee & Cowley (1993). In the two-dimensional case the formulation is expressed in terms of a nonlinear partial differential equation for the vorticity in the critical layer, where the flow speed is close to the wave speed, coupled with a jump condition for the velocity change across the critical layer. The three-dimensional result, on the other hand, is quite different, with the amplitude growth expressed in terms of a nonlinear integro-differential equation.

If the spanwise wavenumber is allowed to approach zero in the formulation of GC or Wu *et al.* it is found that the two-dimensional case is not recovered. The present note introduces a third special limiting case, with a slow spanwise variation. In this limit the spanwise momentum equation expresses a balance between pressure and inertia terms, and the spanwise scale is such that a stretching term appears in the vorticity equation. If this scale is increased or decreased, the results are found to be consistent, respectively, with the two-dimensional formulation or the fully three-dimensional formulation. That is, the results can be said to 'match' asymptotically in terms of differently scaled wavenumber parameters. The intention of this note is only to show the equations needed for this new limiting case, and to indicate briefly how the formulations given in the references are approached when the corresponding length scale increases or decreases.

2. Formulation

We suppose that a laminar shear layer between two parallel uniform streams of incompressible fluid is subjected to a nearly two-dimensional disturbance of small amplitude. The upper and lower streams have velocities $U^{(1)}$ and $U^{(2)}$, respectively, with $U^{(1)} > U^{(2)}$. The rectangular coordinates x, y, and z lie, respectively, in the direction of the undisturbed streams, normal to the shear layer, and along the shear layer in the direction normal to the undisturbed flow. The corresponding velocity components are u, v, and w. The spatial coordinates, the time t, and the velocity components have been made non-dimensional with δ_0, δ_0/Δ, and Δ, respectively, where δ_0 is a shear-layer thickness (which might be related to a momentum thickness or a vorticity thickness at a reference location) and $\Delta = (U^{(1)} - U^{(2)})/2$ is a reference velocity. The Reynolds number $R = \delta_0\Delta/\nu$, where ν is the kinematic viscosity, is taken to be large. Wherever convenient, the notation here and below is taken from that of the references.

The disturbance amplitude in the main part of the shear layer is $O(\varepsilon)$, where $\varepsilon \ll 1$, and the wavenumber component in the x-direction is $\alpha = O(1)$. The disturbance is taken to be nearly neutral, so that the frequency is close to the neutral value S_0 and the wave speed is approximately S_0/α. The undisturbed velocity in the shear layer is $u = U(y)$, taken to be smooth and monotonic, and with the one inflection point located at $y = 0$. At the critical layer, the undisturbed velocity and its derivative are $U_c = S_0/\alpha$ and U_c'; the critical layer is located at $y = 0$ and so $U_c'' = 0$. Solutions are sought in the limit as $\varepsilon \to 0$ and $R \to \infty$, such that the Haberman parameter $\lambda = 1/(\varepsilon^{3/2}R)$ is held fixed. This is the well-known condition for the appearance of both nonlinear and viscous effects in the equations describing the flow in the critical layer; nonlinearity enters because the disturbance occurs near the neutral point. The thickness of the critical layer is $O(\varepsilon^{1/2})$, the frequency $S = S_0 + \varepsilon^{1/2}S_1 + \cdots$ differs from the neutral value by $O(\varepsilon^{1/2})$, and the slow spatial growth of the disturbance occurs on a larger length scale expressed in terms of the slow variable $x_1 = \varepsilon^{1/2}x$. The critical layer is thus a nonlinear, non-equilibrium, viscous critical layer. The appropriate transverse coordinate for the critical layer is $Y = y/\varepsilon^{1/2}$; a coordinate fixed relative to a disturbance moving at the actual wave speed S/α is $\zeta = x - (S/\alpha)t$.

The three-dimensional dependence of the imposed disturbance is assumed to be mild, in the sense that the variation in the z-direction occurs on a large length scale. That is, we consider oblique waves propagating at small angles from the undisturbed flow direction. It is found that the effect of vortex stretching first appears in the critical layer when the spanwise wavelength is $O(\varepsilon^{-1/2})$, and so the appropriate spanwise coordinate is $z_1 = \varepsilon^{1/2}z$. This conclusion can be reached by imagining that we start with the two-dimensional problem and then introduce an extremely slow spanwise variation with scale that is allowed to decrease gradually from infinity until new features appear in the equations. Except for the spanwise coordinate and spanwise velocity, the scales are taken to be the same as for the two-dimensional case. If the scaled spanwise coordinate is βz, where $\beta \ll 1$, the z-momentum equation for the critical layer shows that $w = O(\varepsilon^{1/2}\beta)$ and the vorticity equation shows that a stretching term enters when $w\beta = O(\varepsilon^{3/2})$; it then follows that $\beta = O(\varepsilon^{1/2})$.

In the main part of the shear layer, outside the critical layer, the largest perturbations have the same form as in GL, GH, and H, but now the amplitude function depends on z_1 as well as x_1, and a spanwise velocity component $w = \varepsilon^{3/2}w_1 + \cdots$ is added, where w_1 is found in the form

$$w_1 = \alpha^{-1}\mathrm{Re}\left\{\left[\frac{U'(y)}{U(y)-U_c}\phi_1(y) - \phi_1'(y)\right]\mathrm{i}A_{z_1}(x_1,z_1)\mathrm{e}^{\mathrm{i}\alpha\zeta}\right\}. \tag{1}$$

Here $\phi_1(y)$ is real and satisfies the Rayleigh equation, as in GL, GH, and H; A is a complex amplitude factor that varies on the larger scales of the variables x_1 and z_1; as indicated by the notation Re, the real part is to be taken. Near the critical layer, as $y \to 0$, ϕ_1 has the same form as in H, and $w_1 \sim (\alpha y)^{-1}\mathrm{Re}(\mathrm{i}A_{z_1}\mathrm{e}^{\mathrm{i}\alpha\zeta})$. In the equations for terms of the next order, the coordinate z_1 again enters only as a parameter, and so the equations have the same appearance as for the two-dimensional case. As a consequence, a solvability condition is obtained in the same form as in equations (2.14)–(2.16) of H, the only difference lying in the spanwise variation of the amplitude function $A(x_1, z_1)$.

The expansions for the flow variables within the critical layer also have the same form as in GL, GH, and H, but now with the added velocity component $w = \varepsilon\hat{w}_0 + \cdots$. The solutions for the largest terms $\varepsilon\hat{p}_0$ and $\varepsilon\hat{v}_0$ in the pressure and the transverse velocity component are again just the leading terms in the expansions of the corresponding outer solutions as $y \to \pm 0$. The spanwise component Ω of the vorticity is

$$\Omega = v_x - u_y = -U_c' + \varepsilon\hat{\Omega}_0 + \cdots. \tag{2}$$

The largest terms in the differential equations for w and Ω are then found to give

$$\mathscr{L}\hat{w}_0 = -\hat{p}_{0z_1}, \quad \mathscr{L}\hat{\Omega}_0 = -U_c'\hat{w}_{0z_1}, \tag{3}$$

where

$$\mathscr{L} = U_c\frac{\partial}{\partial x_1} + \left(U_c'Y - \frac{S_1}{\alpha}\right)\frac{\partial}{\partial\zeta} + \hat{v}_0\frac{\partial}{\partial Y} - \lambda\frac{\partial^2}{\partial Y^2}, \tag{4}$$

and $\hat{p}_0 = U_c'\mathrm{Re}(A\mathrm{e}^{\mathrm{i}\alpha\zeta})$, $\hat{v}_0 = -\alpha\mathrm{Re}(\mathrm{i}A\mathrm{e}^{\mathrm{i}\alpha\zeta})$. The stretching term in the vorticity equation of course arises from the variation of the spanwise velocity component in the spanwise direction. As in the references, it is convenient to write the solution for $\hat{\Omega}_0$ in the form

$$\hat{\Omega}_0 = -\tfrac{1}{2}U_c'''Y^2 - \lambda\frac{U_c'''}{U_c}x_1 - \frac{U_c'''}{U_c'}\mathrm{Re}(A(x_1,z_1)\mathrm{e}^{\mathrm{i}\alpha\zeta}) - Q(\zeta, Y, x_1, z_1), \tag{5}$$

where, as in H, the second term on the right-hand side arises from the slow growth of mean shear-layer thickness. The equations for $\hat{w}_0$ and Q become

$$\mathscr{L}\hat{w}_0 = -U_c'\mathrm{Re}(A_{z_1}\mathrm{e}^{\mathrm{i}\alpha\zeta}), \tag{6}$$

$$\mathscr{L}Q = \frac{U_c''}{U_c'}\mathrm{Re}\{(\mathrm{i}S_1A - U_cA_{x_1})\mathrm{e}^{\mathrm{i}\alpha\zeta})\} + U_c'\hat{w}_{0z_1}. \tag{7}$$

As $|Y| \to \infty$, the largest terms in (6) and (7) show that $\hat{w}_0$ and Q are both $O(1/Y)$. The amplitude function A should match upstream, as $x_1 \to -\infty$, with the amplitude predicted by the linear stability theory.

As in GL, GH, and H, the higher harmonics do not enter into the calculation of the flow in the critical layer, but are determined once the critical-layer flow is known.

The function Q is to be multiplied by $e^{-i\alpha\zeta}$, then integrated over one period in ζ and integrated across the critical layer. As in H, the requirement that this result must agree with the jump found from the main part of the shear layer leads to the relation

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{2\pi/\alpha} Q(\zeta, Y, x_1, z_1)e^{-i\alpha\zeta}d\zeta dY = 2iA_{x_1}J_1 - (S_1A + iU_cA_{x_1})J_2, \qquad (8)$$

where, as in H, the constants J_1 and J_2 are defined by

$$J_1 = \int_{-\infty}^{\infty}\phi_1^2 dy, \quad J_2 = \frac{1}{\alpha^2}\int_{-\infty}^{\infty}\frac{U''}{(U-U_c)^2}\phi_1^2 dy \qquad (9)$$

and the integral defining J_2 is interpreted as a principal value. Although the form of (8) is the same as obtained in H for the two-dimensional case, the function Q now depends on z_1 as well as x_1.

3. Discussion

This, then, is the appropriate formulation for a special limiting case corresponding to a specific large spanwise length scale, namely $O(\varepsilon^{-1/2})$. The formulation contains all the terms of the two-dimensional formulation, but the differential equations for the critical layer are augmented by a stretching term in the vorticity equation and by a spanwise momentum equation. It is easy to show that the equations reduce to the two-dimensional formulation if the length scale is allowed to increase. In that case, all derivatives with respect to z_1 approach zero, and the (scaled) spanwise velocity component $\hat{w}_0$ also approaches zero. With these terms omitted, the resulting equations are exactly those for the two-dimensional problem.

In the three-dimensional formulation of GC and Wu *et al.* where the spanwise length scale is $O(1)$, the critical-layer thickness and the frequency perturbation are $O(\varepsilon^{1/3})$, rather than $O(\varepsilon^{1/2})$ as in the two-dimensional case, and the scale for slow streamwise variations is $O(\varepsilon^{-1/3})$ rather than $O(\varepsilon^{-1/2})$. The differential operator analogous to $\mathscr{L}$ in the equations for w and Ω does not include the term containing v. As the spanwise scale becomes large, the GC solution (3.3) for v, with the substitution (3.7), remains $O(\varepsilon)$, while the orders for the critical-layer thickness and frequency perturbation are found to decrease, and the slow streamwise scale increases. When the scale is large enough that the orders of these quantities agree with the orders for two-dimensional disturbances, the contribution of $v\partial/\partial Y$ to the differential operator is no longer of higher order. This is the case when the spanwise scale, inversely proportional to the angle θ in GC, has increased to $O(\varepsilon^{-1/2})$. For this condition, it is also seen that the term v_x in the expansion (3.12) of GC for the vorticity Ω is no longer small in comparison with the perturbation term in u_y. Thus the formulation of GC remains correct when expanded for spanwise length scales that are large, but small in comparison with $\varepsilon^{-1/2}$, and observation of the failure of this formulation provides an alternative, but less transparent, way of determining the special length scale chosen here.

The present formulation can be said to match with the previous formulation in terms of the parameter used to define the spanwise scaling. If the spanwise coordinate is defined as βz, then the present formulation is consistent with that of GC in the limit as $\varepsilon \to 0$ and $\beta \to 0$ with $\beta/\varepsilon^{1/2} \to \infty$ and agrees with the formulation of GL, GH, and H in the limit as $\varepsilon \to 0$ and $\beta \to 0$ with $\beta/\varepsilon^{1/2} \to 0$.

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